On the Lax pairs for the generalized Kowalewski and Goryachev-Chaplygin tops

V. V. Sokolov
Landau Institute for Theoretical Physics,
Kosygina str. 2, Moscow, 117334, Russia
e-mail: sokolov@landau.ac.ru

A. V. Tsiganov
Department of Mathematical and Computational Physics,
St.Petersburg University, St.Petersburg, 198904, Russia
e-mail: tsiganov@mph.phys.spbu.ru

Abstract

A polynomial deformation of the Kowalewski top is considered. This deformation includes as a degeneration a new integrable case for the Kirchhoff equations found recently by one of the authors. A 5×5 matrix Lax pair for the deformed Kowalewski top is proposed. Also deformations of the two-field Kowalewski gyrostat and the $so(p, q)$ Kowalewski top are found. All our Lax pairs are deformations of the corresponding Lax representations found by Reyman and Semenov-Tian Shansky. In addition, a similar deformation of the Goryachev-Chaplygin top and its 3×3 matrix Lax representation is constructed.

1 Introduction.

In the paper [6] it was shown that a top-like system which corresponds to the following Hamilton function

$$H = J_1^2 + J_2^2 + 2J_3^2 - 2(c_2x_1 - a_1)J_3 - 2a_1c_2\,x_1 - c_2^2\,x_3^2 - 2c_1\,x_2\,, \tag{1.1}$$

where c_1, c_2 and a_1 are arbitrary constants, is completely integrable. If $c_2 = a_1 = 0$, then the Hamiltonian just reduces to the famous Kowalewski Hamiltonian. The case $c_2 = 0$ corresponds to the Kowalewski Hamiltonian with the additional gyrostatic term. If $a_1 = c_1$, we get the Hamiltonian function for the integrable case for the Kirchhoff equations found in [5].

It turns out that there exists a polynomial of fourth degree, which commutes with (1.1) with respect to the Lie-Poisson bracket

$$\{J_i\,, J_j\,\} = \varepsilon_{ijk}\,J_k\,, \qquad \{J_i\,, x_j\,\} = \varepsilon_{ijk}\,x_k\,, \qquad \{x_i\,, x_j\,\} = 0\,, \qquad i,j,k = 1,2,3, \tag{1.2}$$

where ε_{ijk} is the standard totally skew-symmetric tensor. These brackets possess two Casimir elements

$$I_1 = (x, x), \qquad I_2 = (J, x)\,, \tag{1.3}$$

where $J = (J_1, J_2, J_3)$, $x = (x_1, x_2, x_3)$ and (x, y) stands for the scalar product in $\mathbb{R}^3$. Since generic symplectic leaves specified by the values of these Casimir elements have dimension four, we need only one additional first integral for the Liouville integrability of the corresponding equations of motion given by the standard formulae

$$J_t = \{J\,, H\,\}, \qquad x_t = \{x\,, H\,\}\,. \tag{1.4}$$

In this paper we present a 5×5-matrix Lax pair for (1.4), (1.1), which generalizes the corresponding Lax pair from the paper [2]. Following [4], we find a deformation of the famous Kowalewski curve with respect to the additional parameter c_2.

We also find generalizations of the two-field gyrostat and the $so(p, q)$-model from [2] and present Lax pairs for them.

Moreover, we find a similar 3×3-matrix Lax pair for a generalized Goryachev-Chaplygin top whose Hamiltonian function has the form

$$H = J_1^2 + J_2^2 + 4J_3^2 - 2a_1 J_3 - 4c_1 x_2 - 4a_1 c_2 x_1 + 8c_2 J_3 x_1 - 4c_2^2 x_3^2. \tag{1.5}$$

If $c_2 = 0$ this Hamiltonian coincides with the usual Goryachev-Chaplygin gyrostat and our Lax pair reduces to the Lax representation from [1]. Like the Goryachev-Chaplygin gyrostat the generalization is an integrable system on the level $I_2 = 0$ only. In the case $a_1 = c_1 = 0$ we get a new partially integrable (i.e. integrable on a special level of one of the integrals of motion) case for the Kirchhoff equations.

2 Generalized Kowalewski top

The Kowalewski gyrostat is defined by the Hamiltonian (1.1) with $c_2 = 0$. In the paper [2] a Lax representation

$$\frac{d}{dt} L_{kow} = [M_{kow},\, L_{kow}]$$

for this system has been found. The corresponding Lax matrices L_{kow} and M_{kow} are given by

$$L_{kow}(\lambda) = \lambda\, A + B + c_1 \lambda^{-1}\, C\,, \qquad M_{kow}(\lambda) = -2\lambda\, A + D \tag{2.6}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & J_3 & -J_2 & 0 & 0 \\ -J_3 & 0 & J_1 & 0 & 0 \\ J_2 & -J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J_3 - a_1 \\ 0 & 0 & 0 & J_3 + a_1 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & -4J_3 - 2a_1 & 2J_2 & 0 & 0 \\ 4J_3 + 2a_1 & 0 & -2J_1 & 0 & 0 \\ -2J_2 & 2J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic curve $Det(L_{kow}(\lambda) - \mu \cdot Id) = 0$, where $Id = diag(1,1,1,1,1)$ is the unit matrix provides a complete set of first integrals for the Kowalewski gyrostat [2].

In this paper we consider a matrix of the following form

$$L(\lambda,\mu) = L_1(\lambda) + \mu \cdot L_2(\lambda), \tag{2.7}$$

where

$$L_1(\lambda) = L_{kow}(\lambda) + c_2 X, \qquad L_2(\lambda) = -Id + c_2\lambda^{-1} Y,$$

and

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & x_1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & -x_3 & 0 & 0 \end{pmatrix}.$$

Obviously, if $c_2 = 0$ then this matrix $L(\lambda,\mu)$ coincides with $L_{kow}(\lambda) - \mu \cdot Id$.

It is easy to verify that for operator (2.7) the following symmetry properties hold:

$$L(\lambda,\mu) = -L^T(-\lambda,-\mu), \qquad L(\lambda,\mu) = V^{-1}L(-\lambda,\mu)V, \tag{2.8}$$

where $V = diag(1,1,1,-1,-1)$.

A simple calculation shows that the algebraic curve $\mathcal{C}$: $Det\Big(L(\lambda,\mu)\Big) = 0$ can be written in the following form

$$\mathcal{C}: \quad d_2(\lambda^2)\,\mu^4 + d_1(\lambda^2)\,\mu^2 + d_0(\lambda^2) = 0,$$

$$d_2 = \lambda^2 + c_2^2 I_1, \qquad d_0 = \lambda^6 - H\lambda^4 + I_4\lambda^2 - c_1^2 I_2^2, \tag{2.9}$$

$$d_1 = -2\lambda^4 + (H + a_1^2 - c_2^2 I_1)\lambda^2 + (c_2^2 I_2^2 - c_1^2 I_1),$$

where

$$H = J_1^2 + J_2^2 + 2J_3^2 - 2c_1x_2 + 2a_1J_3 + 2c_2(J_3x_1 - x_3J_1) \tag{2.10}$$

and

$$\begin{aligned} I_4 &= x_1\Big(x_1(J_2^2 + J_3^2 - J_1^2) - 2(x_3J_3 + x_2J_2)J_1\Big)c_2^2 + (x_2^2 + x_3^2)c_1^2 \\ &+ 2\Big(c_1x_1(x_3J_2 - x_2J_3) + (J_3 + a_1)(x_1(J_2^2 + J_3^2) - (x_2J_2 + x_3J_3)J_1)\Big)c_2 \\ &- 2\Big(x_2(J_2^2 + J_3^2 + a_1J_3) + (x_1J_1 - a_1x_3)J_2\Big)c_1 + (J_1^2 + J_2^2 + J_3^2)(J_3 + a_1)^2. \end{aligned} \tag{2.11}$$

The following statement can be proved by a straightforward calculation.

Proposition 1 $\{H,\, I_4\} = 0.$

It follows from Proposition 1 that the functions $I_3 = H$ and I_4 are integrals of motion in involution and the corresponding Hamiltonian system is completely integrable. Notice that the Hamiltonian (2.10) up to a canonical transformation of the form

$$J_1 \to J_1 + c_2x_3, \qquad J_2 \to J_2, \qquad J_3 \to J_3 - c_2x_1,$$

coincides with (1.1).

The next theorem describes Lax structures related to the operator (2.7).

Theorem 1 *The flow with the Hamiltonian (2.10) is equivalent to the following matrix differential equations*

$$\frac{d}{dt}L_i = L_i\, M(\lambda) + M^T(-\lambda)\, L_i, \qquad i=1,2, \tag{2.12}$$

where

$$M = M_{kow} + W, \qquad W = 2c_2 \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & -x_2 \end{pmatrix}, \tag{2.13}$$

and the superscript T stands for matrix transposition.

The relations (2.12) imply that the operators

$$L_+ = L_1(\lambda)\, L_2^{-1}(\lambda), \qquad L_- = L_2^{-1}(\lambda)\, L_1(\lambda) \tag{2.14}$$

satisfy the usual Lax equations

$$\frac{d}{dt}L_+ = \big[L_+,\, -M^T(-\lambda)\big]\,, \quad \frac{d}{dt}L_- = [L_-,\, M(\lambda)]\,. \tag{2.15}$$

An explicit formula for L_2^{-1} can be written as follows:

$$L_2^{-1} = -\Big(Id + \frac{1}{\lambda}Y + \frac{1}{\lambda^2}Y^2 + \frac{1}{\lambda^3\Delta}Y^3 + \frac{1}{\lambda^4\Delta}Y^4\Big),$$

where

$$\Delta = -Det(L_2) = 1 + \frac{c_2^2 I_1}{\lambda^2}$$

is a Casimir function.

It is clear that the determinant curves $Det(L(\lambda,\mu)) = 0$, $Det(L_+(\lambda) - \mu Id) = 0$, and $Det(L_-(\lambda) - \mu Id) = 0$ coincide with each other up to inessential multipliers. Therefore one can use one of the operators $L_\pm$ to generalize the results of [2], though to our taste the operator L looks more elegant than the operators $L_\pm$.

Remark. The Lax triads (2.12) and the Lax matrices of the form $L_1(\lambda)L_2^{-1}(\lambda)$ have arisen in the case of the relativistic Toda lattice (see [3] and references within). However in that case the operator L_1 was the same for the initial and deformed models whereas in our case L_1 must be deformed also.

According to [2] let us consider the projection of curve C (2.9) given by the change of variables $z = \lambda^2$

$$\mathcal{C}_1: \qquad d_2(z)\mu^4 + d_1(z)\mu^2 + d_0(z) = 0\,.$$

The genus of $\mathcal{C}_1$ reduces from 3 to 2 in the following two cases: $a_1 = I_2 = 0$ or $a_1 = c_1 = 0$. The latter case corresponds to the new integrable case for the Kirchhoff equations found in [5].

Let us consider the first case. Following [4], we can easily find a deformation of the famous Kowalewski curve with respect to the parameter c_2. Namely, the following transformation

$$\mu = \frac{y}{x^2 + (H + I_1c_2^2)\,x + I_4 - I_1c_1^2}, \qquad z = \frac{I_1x(c_1^2 - c_2^2x)}{x^2 + (H + I_1c_2^2)\,x + I_4 - I_1c_1^2} \tag{2.16}$$

brings $\mathcal{C}_1$ to the normal form

$$\mathcal{C}_2: \qquad y^2 = x\left(x^2 + Hx + I_4\right)\left(x^2 + (H + c_2^2 I_1)x + I_4 - c_1^2 I_1\right) . \tag{2.17}$$

This curve differs from the corresponding curve from [2] by the third factor only. According to [4] we can use Richelot's transformation of the curve $\mathcal{C}_2$ in order to get another hyperelliptic curve

$$\widetilde{\mathcal{C}}_2: \qquad \eta^2 = \Bigl(c_2^2\zeta^2 - 2c_1^2\zeta - Hc_1^2 - I_4c_2^2\Bigr)\Bigl(\zeta^2 - I_4 + I_1c_1^2\Bigr)\Bigl(\zeta^2 - I_4\Bigr) \tag{2.18}$$

which is a deformation of the usual Kowalewski curve. As above, the genus of this curve is not changed and there is a difference in the first factor only. Of course, if $c_2 = 0$ and $c_1 = 1$ then the curve $\widetilde{\mathcal{C}}_2$ coincides with the Kowalewski curve [4].

In the second case the normal form of $\mathcal{C}_1$ is

$$\mathcal{C}_3: \qquad y^2 = x\left(x^2 + Hx + I_4\right)\left(x^2 + (H + c_2^2 I_1)x + I_4 + c_2^2 I_2^2\right) . \tag{2.19}$$

and Richelot's transformation gives rise to

$$\widetilde{\mathcal{C}}_3: \qquad \eta^2 = \Bigl(I_1\zeta^2 + 2I_2^2\zeta + HI_2^2 - I_4I_1\Bigr)\Bigl(\zeta^2 - I_4 - I_2^2c_2^2\Bigr)\Bigl(\zeta^2 - I_4\Bigr) \tag{2.20}$$

It is interesting to note a duality between I_2 and c_1 in Case 1 and Case 2. By analogy with the Kowalewski case one can expect that (2.18) and (2.20) are separation curves for the corresponding cases.

3 Generalized two-field gyrostat

In the two-field case we have three vectors $J = (J_1, J_2, J_3)$, $x = (x_1, x_2, x_3)$, and $y = (y_1, y_2, y_3)$. The Lie-Poisson bracket is given by

$$\begin{aligned} &\{J_i\,, J_j\,\} = \varepsilon_{ijk}\, J_k\,, \qquad \{J_i\,, x_j\,\} = \varepsilon_{ijk}\, x_k\,, \qquad \{x_i\,, x_j\,\} = 0 \\ &\{J_i\,, y_j\,\} = \varepsilon_{ijk}\, y_k\,, \qquad \{y_i\,, y_j\,\} = 0, \qquad \{x_i\,, y_j\,\} = 0, \qquad i,j,k = 1,2,3. \end{aligned} \tag{3.21}$$

The Casimir functions are $(x,\, x)$, $(x,\, y)$, and $(y,\, y)$.

We claim that the Hamiltonian function

$$H = J_1^2 + J_2^2 + 2J_3^2 - 2c_1x_2 - 2b_1y_1 + 2a_1J_3 + 2c_2(J_3x_1 - x_3J_1) - 2b_2(J_3y_2 - J_2y_3) \tag{3.22}$$

gives rise to a completely integrable model if

$$c_1b_2 - b_1c_2 = 0.$$

Although the parameters can be normalized by scalings, we prefer to keep them because all reductions and limits are more obvious in this form. In the case $b_1 = b_2 = 0$ the Hamiltonian function (3.22) coincides with (2.10). If $a_1 = c_1 = b_1 = 0$ we have a new homogeneous quadratic integrable Hamiltonian.

Two necessary additional integrals of motion are the coefficients at λ^4 and λ^2 of the algebraic curve $Det\Bigl(L(\lambda,\mu)\Bigr) = 0$, where $L = L_1 + \mu L_2$,

$$L_1 = \lambda A + B + \widehat{X} + \lambda^{-1}\widehat{C}, \qquad L_2 = -Id + \lambda^{-1}\widehat{Y}.$$

The matrices A and B are defined in the previous section and

$$\widehat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_2x_1 + b_2y_2 \\ 0 & 0 & 0 & c_2x_1 - b_2y_2 & 0 \end{pmatrix},$$

$$\widehat{C} = \begin{pmatrix} 0 & 0 & 0 & b_1y_1 & c_1x_1 \\ 0 & 0 & 0 & b_1y_2 & c_1x_2 \\ 0 & 0 & 0 & b_1y_3 & c_1x_3 \\ b_1y_1 & b_1y_2 & b_1y_3 & 0 & 0 \\ c_1x_1 & c_1x_2 & c_1x_3 & 0 & 0 \end{pmatrix}, \qquad \widehat{Y} = \begin{pmatrix} 0 & 0 & 0 & b_2y_1 & c_2x_1 \\ 0 & 0 & 0 & b_2y_2 & c_2x_2 \\ 0 & 0 & 0 & b_2y_3 & c_2x_3 \\ -b_2y_1 & -b_2y_2 & -b_2y_3 & 0 & 0 \\ -c_2x_1 & -c_2x_2 & -c_2x_3 & 0 & 0 \end{pmatrix}.$$

The operators L_1 and L_2 satisfy (2.8) and (2.12), with

$$M = M_{kow} + \widehat{W}, \qquad \widehat{W} = 2\begin{pmatrix} b_2y_1 & c_2x_1 & 0 & 0 & 0 \\ b_2y_2 & c_2x_2 & 0 & 0 & 0 \\ b_2y_3 & c_2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2y_1 & -c_2x_1 \\ 0 & 0 & 0 & -b_2y_2 & -c_2x_2 \end{pmatrix}. \tag{3.23}$$

4 Generalized q-field top

In this Section we present an integrable polynomial deformation of the $so(p, q)$ Kowalewski system. If $p = 3$ and $q = 2$, then the deformed Hamiltonian coincides with (2.10), where $a_1 = 0$, $c_1 = c_2 = 1$.

Recall that for any p and q such that $q \le p$ the Hamiltonian

$$H_{old} = \frac{1}{2}\Big(\sum_{i,j=1}^{p} l_{ij}^2 + \sum_{i,j=1}^{q} l_{ij}^2\Big) - 2\sum_{i=1}^{q} F_{ii}\,.$$

defines the so called $so(p, q)$- analog of the Kowalewski top. Here c is arbitrary constant, and dynamical variables l_{ij} and F_{ij} are entries of a skew-symmetric $p \times p$ matrix l and a $p \times q$ matrix F.

The $(p + q) \times (p + q)$ matrix Lax pair for this system found in [2] is given by

$$L_{old}(\lambda) = \lambda\, A + B + \lambda^{-1}\, C\,, \qquad M_{old}(\lambda) = -2\lambda\, A + D$$

where

$$A = \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -l & 0 \\ 0 & E^T l E \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}.$$

Here E is a $p \times q$ matrix, whose non-zero entries are $E_{ii} = 1,\ i \le q$. The entries of the matrix ω are defined as follows: $\omega_{ij} = 4l_{ij}$, if $i, j \le q$ and otherwise $\omega_{i,j} = 2l_{ij}$.

Let us consider a deformation of this Hamiltonian

$$H = H_{old} - 2\sum_{i=1}^{p}\sum_{j=1}^{q} F_{ij} l_{ij}\,.$$

The corresponding Hamiltonian flow is equivalent to the matrix differential equations (2.12) with matrices

$$L(\lambda,\mu)=L_1(\lambda)+\mu L_2(\lambda)\,,\qquad M(\lambda)=M_{old}(\lambda)+W\,,$$

where

$$L_1(\lambda)=L_{old}+X\,,\qquad L_2(\lambda)=-Id+\lambda^{-1}Y$$

and

$$Y=\begin{pmatrix}0 & F\\ -F^T & 0\end{pmatrix},\quad X=\frac{1}{2}\Big((C-Y)A-A(C+Y)\Big),\qquad W=\Big((C+Y)A-A(C+Y)\Big).$$

5 Generalized Goryachev-Chaplygin top

In this section we establish analogous structures for the simpler case of the Goryachev-Chaplygin top.

Let us consider the following 3×3 matrix $L(\lambda,\mu)=L_1(\lambda)+\mu L_2(\lambda)$, where

$$L_1=\lambda S+J+c_2B+ic_1\lambda^{-1}\,X,\qquad L_2=Id-c_2\lambda^{-1}\,Y, \tag{5.24}$$

and

$$S=\begin{pmatrix}0&0&0\\0&0&-2\\0&2&0\end{pmatrix},\qquad J=\begin{pmatrix}0&0&-J_1-iJ_2\\0&-2J_3+a_1&0\\-J_1+iJ_2&0&2J_3\end{pmatrix}$$

$$X=\begin{pmatrix}0&x_3&0\\x_3&0&x_1+ix_2\\0&x_1-ix_2&0\end{pmatrix},\qquad Y=\begin{pmatrix}0&x_3&0\\-x_3&0&-x_1-ix_2\\0&x_1-ix_2&0\end{pmatrix},$$

$$B=\frac{1}{2}\Big((X-Y)S-S(X+Y)\Big)=4x_1\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

Everywhere we assume that $I_2=0$. If $c_2=0$, this L-operator coincides with the operator found in [1].

The corresponding spectral curve is

$$(\lambda^2+c_2^2I_1)\,\mu^3-a_1\lambda^2\mu^2+(4\lambda^4-H\lambda^2+c_1^2I_1)\mu-I_4\lambda^2=0$$

where

$$H \;=\; J_1^2+J_2^2+4J_3^2-2a_1J_3-4c_1x_2+4c_2(J_1x_3-2J_3x_1)\,, \tag{5.25}$$

$$I_4 \;=\; (J_1^2+J_2^2)(2J_3-4c_2x_1-a_1)+4c_1J_2x_3\,. \tag{5.26}$$

It is easy to verify that, for the operator L, the following symmetry properties hold:

$$L^*(-\lambda,\mu)=L(\lambda,\mu),\qquad L(-\lambda,\mu)=VL(\lambda,\mu)V^{-1}, \tag{5.27}$$

where $V=diag(1,-1,1)$ and $*$ means Hermitian conjugation.

Theorem 2 *The flow with the Hamiltonian (5.25) is equivalent to the following matrix differential equations*

$$\frac{d}{dt} L_i = L_i\, M(\lambda) + M^*(-\lambda)\, L_i, \qquad i = 1, 2, \tag{5.28}$$

where $M = 2i(\lambda S + W)$,

$$W = \begin{pmatrix} -J_3 & 0 & -J_1 - iJ_2 \\ 0 & 0 & 0 \\ -J_1 + iJ_2 & 0 & 4J_3 - a_1 \end{pmatrix} - 2c_2 \begin{pmatrix} 0 & 0 & x_3 \\ 0 & ix_2 & 0 \\ 0 & 0 & 2x_1 - ix_2 \end{pmatrix}.$$

After the canonical transformation

$$J_1 \to J_1 - 2c_2 x_3, \qquad J_2 \to J_2, \qquad J_3 \to J_3 + 2c_2 x_1,$$

this Hamilton function takes the form (1.5).

Acknowledgments. The authors are grateful to the Newton Institute (Univ. of Cambridge) for its hospitality. The research was partially supported by RFBR grants 99-01-00294 and 99-01-00698, INTAS grants 99-1782 and 99-01459, and EPSRC grant GR K99015.

References

[1] A.I. Bobenko and V.B. Kuznetsov, Lax representation and new formulae for the Goryachev-Chaplygin top, *J.Phys.A.*, v. 21, p. 1999, 1988.

[2] A.G. Reyman and M.A. Semenov-Tian Shansky, Lax representation with a spectral parameter for the Kowalewski top and its generalizations, *Lett.Math.Phys*, v.14, p.55, 1987.

A.I. Bobenko and A.G. Reyman and M.A. Semenov-Tian Shansky, The Kowalewski top 99 years later: a Lax pair, generalizations and explicit solutions, *Commun.Math.Phys.*, v.122, p.321, 1989.

[3] Yu. B. Suris, On the bi-Hamiltonian structure of Toda and relativistic Toda lattices, *Phys. Lett. A.*, v.180, p.419, 1993.

[4] D. Markushevich, Kowalewski top and genus-2 curves, *J.Phys.A.*, v.34, p.2125, 2001.

[5] V.V. Sokolov, A new integrable case for the Kirchhoff equation, *Teor.Math.Phys.*, v.128(2), p.31, 2001.

[6] V.V. Sokolov, A generalized Kowalevski Hamiltonian and new integrable cases on $e(3)$ and $so(4)$, Preprint *nlin.SI/0110022*, 2001.